Further Extension of Fan-Browder Coincidence and Fixed Point Theorems

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Introduction

A Simple Extension of Fan-Browder Coincidence Theorem

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I Introduction

The fixed-point and coincidence theorems of F. Browder and K. Fan (see Browder (1968) and Fan (1969)) are well known and frequently used in mathematical economics and game-theoretic equilibrium arguments. A series of their results are based on the next famous fixed-point theorem known as Browder’s fixed point theorem:

**Theorem 1**: (Browder 1968, Theorem 1). Let $X$ be a non-empty compact convex subset of a Hausdorff topological vector space $E$. Let $T: X \rightarrow X$ be a non-empty and convex valued correspondence. Suppose further that for all $y \in X$, $T^{-1}(y) \equiv \{ x \in X \mid y \in T(x) \}$ is open in $X$. Then $T: X \rightarrow X$ has a fixed point $x^* \in X$.

Note that in the above the vector space, $E$, is assumed to be Hausdorff, but not necessarily to be locally convex.

In the context of economics, the vector space duality is often interpreted as the duality between commodities and prices. Some natural assumptions on agents’ behaviors in response to prices would therefore enable us to utilize rather weak topological conditions on the commodity space to ensure the existence of an equilibrium. From this motivation, the authors have studied several extensions of Kakutani’s fixed-point theorem and their applications to the economic equilibrium theory (see Urai and Hayashi (2000), Urai (2000), Urai and Yoshimachi

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In this paper, we provide several extensions of the coincidence and fixed-point theorems in Browder (1968) and Fan (1969). Our proofs depend merely on Browder’s fixed-point theorem as given above. Moreover, it can be seen that our results directly show one of the most general kinds of Kakutani’s fixed point theorem on Hausdorff topological vector spaces.

II A Simple Extension of Fan-Browder Coincidence Theorem

First, we directly show a simple extension of the Fan-Browder coincidence theorem by applying Broder’s fixed-point theorem. This holds in the special case that the range of the correspondence $x \mapsto x + (F(x) - G(x))$ happens to be a subset of its domain.

**Theorem 2**: (A Simple Extension of Fan-Browder Coincidence Theorem). Let $X$ be a non-empty compact convex subset of a Hausdorff topological vector space $E$ over $R$ and $F : X \rightarrow E$ and $G : X \rightarrow E$ be two non-empty valued correspondences. Assume the following two conditions:

(i) For each $x \in X$, $x + (F(x) - G(x)) \subseteq X$;

(ii) For each $x$ such that $0 \notin F(x) - G(x)$, element $p_x$ in topological dual $E'$ of $E$ and open neighborhood $U(x)$ of $x$ exist such that $\langle p_x, y \rangle > 0$ for all $y \in F(z) - G(z)$ and $z \in U_x$.

Then there exists a point $x^* \in X$ such that $F(x^*) \cap G(x^*) \neq \emptyset$.

**Proof**: Assume that for each $x \in X$, $F(x) \cap G(x) = \emptyset$, so that $0 \notin F(x) - G(x)$. By condition (i), we can define a non-empty valued correspondence $\varphi : X \rightarrow X$ as $\varphi(x) = x + (F(x) - G(x))$. Note that for all $x \in X$, $x \notin \varphi(x)$. Take any $x \in X$. By condition (ii), there exist an open neighborhood $U(x)$ of $x$ and an element $p_x \in E'$ such that $\langle p_x, z \rangle > 0$ holds for all $x' \in U(x)$ and $z \in F(x') - G(x')$ ($= \varphi(x') - x'$). Thus $\{ U(x) \mid x \in X \}$ is an open covering of $X$, so it has a finite subcovering $\{ U(x_1), \ldots, U(x_n) \}$. Let $\{ \alpha_i : [0, 1] \mid i = 1, \ldots, n \}$ be a partition of unity subordinate to $\{ U(x_1), \ldots, U(x_n) \}$. Define a correspondence $\Psi : X \rightarrow X$ as $\Psi(x) = \{ x + z \in X \mid \sum_{i=1}^{n} \alpha_i(x) \langle p_{x_i}, z \rangle > 0 \}$ for each $x \in X$. It is clear that $\varphi(x) \subseteq \Psi(x)$, hence in particular $\Psi(x) \neq \emptyset$, and $\Psi(x)$ is convex for all $x \in X$. Moreover, $\Psi'(y)$ is open in $X$ for all $y \in X$. Indeed, for any $x \in \Psi'(y)$, we have that $y = x + z$ and $\sum_{i=1}^{n} \alpha_i(x) \langle p_{x_i}, z \rangle > 0$ or, equivalently, $\sum_{i=1}^{n} \alpha_i(x) \langle p_{x_i}, y \rangle - \sum_{i=1}^{n} \alpha_i(x) \langle p_{x_i}, x \rangle > 0$. Since the left-hand side of this inequality is
continuous, there exists a neighborhood $V$ of $x$ such that $x' \in V = \sum_{i=1}^{n} \alpha_i(x') (p_i, y) - \sum_{i=1}^{n} \alpha_i(x') (p_i, x') > 0$, i.e., $V \subset \Psi(y)$. Therefore, by Browder's fixed point theorem, $\Psi$ has a fixed point $x^* \in X$. This means that $\sum_{i=1}^{n} \alpha_i(x^*) (p_i, x^* - x^*) > 0$, a contradiction. 

The above special case immediately implies the following generalization of Kakutani's fixed-point theorem in Urai and Hayashi (2000):

**Theorem 3:** (Urai and Hayashi 2000, Theorem 4). Let $X$ be a non-empty compact convex subset of a Hausdorff topological vector space $E$ over $R$ and $F : X \to X$ be a non-empty valued correspondence. Assume the following condition:

(K) For each $x$ such that $x \not\in F(x)$, element $p_i$ in topological dual $E'$ of $E$ and open neighborhood $U(x)$ of $x$ exist such that $\langle p_i, y \rangle > 0$ for all $y \in F(z) - z$ and $z \in U(x)$.

Then fixed-point $x^* \in X$ of $F$, $x^* \in F(x^*)$, exists.

**Proof:** In Theorem 2, let $G : X \to E$ be the identity map. Then condition (i) in Theorem 2 is automatically satisfied. It is also easy to check that condition (K) for $F$ implies condition (ii) in Theorem 2, and $x^* \in F(x) \cap G(x)$ is nothing but the fixed-point of $F$. 

In this theorem (as not in the theorem but in the proof of Urai and Hayashi (2000, Theorem 4)), the topological vector space $E$ need not be locally convex.

### III Extensions of Fan-Browder Coincidence Theorems

Next, we show an extension of the Fan-Browder coincidence theorem through the following two theorems, Theorem 4 and Theorem 5. (Theorem 4 is an abstract version of Theorem 2 of Browder(1968); Theorems 5 and 6 are extensions of Theorems 5 and 6 of Fan (1969).)

**Theorem 4:** Let $X$ be a non-empty compact convex subset of a Hausdorff topological vector space $E$ over $R$, and $\Psi$ be a (single-valued) mapping from $X$ to $E'$ (the topological dual of $E$). If $f(x) \stackrel{\text{def}}{=} \langle \Psi(x), y-x \rangle$ is a continuous function of $x$ on $X$ for each fixed $y$ in $X$, then there exists a point $x_0 \in X$ such that

$\langle \Psi(x_0), y-x_0 \rangle \leq 0$ for all $y \in X$.

**Proof:** Suppose that the assertion is false. Then for each $x_0 \in X$ there exists an element $y \in X$
such that \( \langle \Psi(x_0), y - x_0 \rangle > 0 \). Let \( T: X \rightarrow X \) be a correspondence defined as

\[
T(x_0) \overset{\text{def.}}{=} \{ y \in X \mid \langle \Psi(x_0), y - x_0 \rangle > 0 \}, \quad \text{where } x_0 \in X.
\]

Note that \( T(x_0) \neq \emptyset \) for each \( x_0 \in X \) by the preceding remark, and obviously \( T(x_0) \) is convex for all \( x_0 \in X \). Moreover, \( T^{-1}(y) \overset{\text{def.}}{=} \{ x \in X \mid y \in T(x) \} \) is open in \( X \) for all \( y \in X \) from the assumption of this theorem. Indeed, \( x \in T^{-1}(y) \implies y \in T(x) \implies \langle \Psi(x), y - x \rangle > 0 \implies \exists U(x) : \) a neighborhood of \( x \) s.t. \( \langle \Psi(x'), y - x' \rangle > 0 \) for all \( x' \in U(x) \implies y \in T(x') \) for all \( x' \in U(x) \implies U(x) \subseteq T^{-1}(y) \).

By Browder’s fixed point theorem, there exists an element \( x^* \in X \) such that \( x^* \in T(x^*) \). This means that \( 0 < \langle \Psi(x^*), x^* - x^* \rangle = \langle \Psi(x^*), 0 \rangle = 0 \), a contradiction. \( \blacksquare \)

**Theorem 5**: Let \( X \) be a non-empty compact convex subset of a Hausdorff topological vector space \( E \) over \( R \), and \( F: X \rightarrow E \) and \( G: X \rightarrow E \) be two non-empty valued correspondences. Assume the following two conditions:

(i) For each \( x \in X \), \( \cup_{\lambda \geq 0} (x + \lambda (F(x) - G(x))) \cap X \neq \emptyset \);

(ii) For each \( x \) such that \( F(x) \) and \( G(x) \) can be strictly separated by a closed hyperplane, there exist an element \( p_\alpha \) in topological dual \( E' \) of \( E \) and an open neighborhood \( U(x) \) of \( x \) such that \( \langle p_\alpha, y \rangle > 0 \) for all \( y \in F(z) - G(z) \) and \( z \in U(x) \).

Then there exists a point \( x^* \in X \) for which \( F(x^*) \) and \( G(x^*) \) cannot be strictly separated by a closed hyperplane.

**Proof**: Suppose that for all \( x \in X \), \( F(x) \) and \( G(x) \) are strictly separated by a closed hyperplane. By condition (ii), there exist \( p_\alpha \in E' \) and an open neighborhood \( U(x) \) of \( x \) such that \( \langle p_\alpha, y \rangle > 0 \) for all \( y \in F(z) - G(z) \) and \( z \in U(x) \). Since \( \{ U(x) \}_{x \in X} \) is an open covering of a compact set \( X \) it has a finite subcovering \( \{ U(x_i) \}_{i=1}^n \). Let \( \{ \alpha_i : X \rightarrow [0, 1] \mid i = 1, \ldots, n \} \) be a partition of unity subordinate to \( \{ U(x_i), \ldots, U(x_n) \} \) and define a single-valued mapping \( \Psi: X \rightarrow E' \) as \( \Psi(x) \overset{\text{def.}}{=} \sum_{i=1}^n \alpha_i(x) p_\alpha \) for each \( x \in X \). Then

\[ \langle \Psi(x), y \rangle > 0 \quad \text{for all } x \in X \text{ and } y \in F(x) - G(x), \]  \( \quad \text{(1)} \)

or equivalently

\[ \langle \Psi(x), u - v \rangle > 0 \quad \text{for all } x \in X, u \in F(x), \text{ and } v \in G(x), \]  \( \quad \text{(2)} \)

since \( \alpha(x) > 0 \implies x \in U(x) \implies \langle p_\alpha, y \rangle > 0 \) for \( y \in F(x) - G(x) \).
Note that $\langle \Psi(x), y - x \rangle = \sum_{i=1}^{n} \alpha_i(x)\langle p_i, y - x \rangle$ and hence $f(x) \overset{\text{def.}}{=} \langle \Psi(x), y - x \rangle$ is a continuous function of $x$ on $X$ for each fixed $y$ in $X$. By Theorem 4, there exists an element $x_0 \in X$ such that

$$\langle \Psi(x_0), y - x_0 \rangle \leq 0 \text{ for all } y \in X.$$  

(3)

By condition (i), there exist three points $u_0 \in F(x_0)$, $v_0 \in G(x_0)$, $y_0 \in X$ and a real number $\lambda > 0$ such that $x_0 + \lambda (u_0 - v_0) = y_0$ or, equivalently, $u_0 - v_0 = \frac{1}{\lambda}(y_0 - x_0)$. Then by (3) we have

$$\langle \Psi(x_0), u_0 - v_0 \rangle = \frac{1}{\lambda} \langle \Psi(x_0), y_0 - x_0 \rangle \leq 0$$

(4)

where $x_0 \in X$, $u_0 \in F(x_0)$, and $v_0 \in G(x_0)$. This contradicts (2).

Now, an extension of the Fan-Browder coincidence theorem follows directly as an application of Theorem 5.

**Theorem 6**: (The First Extension of the Fan-Browder Coincidence Theorem). Let $X$ be a non-empty compact convex set in a locally convex, Hausdorff topological vector space $E$ over $R$. Let $F : X \to E$ and $G : X \to E$ be two non-empty valued correspondences. Assume the following conditions:

(i) For each $x \in X$, there exist three points $y \in X$, $u \in F(x)$, $v \in G(x)$ and a real number $\lambda > 0$ such that $y - x = \lambda (u - v)$;

(ii) For each $x$ such that $F(x)$ and $G(x)$ can be strictly separated by a closed hyperplane, element $p_x$ in topological dual $E'$ of $E$ and open neighborhood $U(x)$ of $x$ exist such that $\langle p_x, y \rangle > 0$ for all $y \in F(z) - G(z)$ and $z \in U(x)$;

(iii) For each $x \in X$, $F(x)$ and $G(x)$ are closed convex sets in $E$ and at least one of them is compact.

Then there exists a point $x^* \in X$ for which $F(x^*)$ and $G(x^*)$ have a non-empty intersection.

**Proof**: Note that all the assumptions of the previous theorem are satisfied here; in particular, condition (i) of both theorems are equivalent. Hence, the theorem follows from the fact that in a locally convex space, two disjoint closed convex sets of which at least one is compact can be strictly separated by a closed hyperplane.

In Theorem 6, as well as in Theorem 5, condition (ii) is weaker than assuming the upper
Theorem 2: In Theorem 6 for the relation between 0 and \(F\) and \(G\). It would also be possible to directly assume a condition like (ii) in Theorem 6 for the relation between 0 and \(F(x) - G(x)\), as in our simple extension given by Theorem 2.

**Theorem 7:** (The Second Extension of the Fan-Browder Coincidence Theorem). Let \(X\) be a non-empty compact convex set in a Hausdorff topological vector space \(E\) over \(R\). Let \(F: X \rightarrow E\) and \(G: X \rightarrow E\) be two non-empty valued correspondences. Assume the following two conditions:

(i) For each \(x \in X\), there exist three points \(y \in X\), \(u \in F(x)\), \(v \in G(x)\) and a real number \(\lambda > 0\) such that \(y - x = \lambda (u - v)\);

(ii) For each \(x\) such that \(0 \notin F(x) - G(x)\), there exist an element \(p\), in the topological dual \(E^*\) of \(E\) and an open neighborhood \(U(x)\) of \(x\) such that \(\langle p, y \rangle > 0\) for all \(y \in F(z) - G(z)\) and \(z \in U(x)\).

Then there exists a point \(x^* \in X\) for which \(F(x^*)\) and \(G(x^*)\) have a non-empty intersection.

**Proof:** Suppose that for all \(x \in X\), \(F(x) \cap G(x) = \emptyset\), and hence, \(0 \notin F(x) - G(x)\). Then by condition (ii) we have \(p \in E^*\) and open neighborhood \(U(x)\) of \(x\) such that \(\langle p, y \rangle > 0\) for all \(y \in F(z) - G(z)\) and \(z \in U(x)\). Since \(\{U(x)\}_{x \in X}\) is an open covering of \(X\), a finite subcovering \(\{U(x_i)\}_{i=1}^n\) exists. Let \(\{\alpha_i: X \rightarrow [0, 1] \mid i = 1, \ldots, n\}\) be a partition of unity subordinate to \(\{U(x_i), \ldots, U(x_n)\}\) and define single-valued mapping \(\Psi: X \rightarrow E^*\) as \(\Psi(x) \triangleq \sum_{i=1}^n \alpha_i(x)p_i\) for each \(x \in X\). Then

\[
\langle \Psi(x), y \rangle > 0 \quad \text{for all } x \in X \text{ and } y \in F(x) - G(x),
\]

or, equivalently,

\[
\langle \Psi(x), u - v \rangle > 0 \quad \text{for all } x \in X, u \in F(x), \text{ and } v \in G(x),
\]

since \(\alpha_i(x) > 0 \quad \Rightarrow \quad x \in U(x_i) \quad \Rightarrow \quad \langle p_i, y \rangle > 0 \quad \text{for } y \in F(x) - G(x)\).

Note that \(\langle \Psi(x), y - x \rangle = \sum_{i=1}^n \alpha_i(x)\langle p_i, y - x \rangle\) and hence \(f(x) \triangleq \langle \Psi(x), y - x \rangle\) is a continuous function of \(x\) on \(X\) for each fixed \(y\) in \(X\). By Theorem 4, there exists an element \(x_0 \in X\) such that

\[
\langle \Psi(x_0), y - x_0 \rangle \leq 0 \quad \text{for all } y \in X.
\]

By condition (i), there exist three points \(u_0 \in F(x_0), v_0 \in G(x_0), y_0 \in X\) and a real number \(\lambda > 0\)
such that \( x_0 + \lambda (u_0 - v_0) = y_0 \) or, equivalently, \( u_0 - v_0 = \frac{1}{\lambda} (y_0 - x_0) \). Then by (7) we have

\[
\langle \Psi(x_0), u_0 - v_0 \rangle = \frac{1}{\lambda} \langle \Psi(x_0), y_0 - x_0 \rangle \leq 0
\]

where \( x_0 \in X, u_0 \in F(x_0), \) and \( v_0 \in G(x_0) \). This contradicts (6).

As Theorem 2 gives one of the most general extensions of Kakutani’s fixed-point theorem, we could also use Theorem 7 to obtain an extension of the Urai-Hayashi fixed-point theorem.

**Theorem 8**: (An Extension of the Urai-Hayashi Fixed-Point Theorem). Let \( X \) be a non-empty compact convex subset of a Hausdorff topological vector space \( E \) over \( R \), and \( F : X \rightarrow E \) be a non-empty valued correspondence. Assume the following conditions:

(i) For each \( x \) such that \( x \not\in F(x) \), two points \( y \in X \) and \( u \in F(x) \), and a real number, \( \lambda > 0 \), exist such that \( y - x = \lambda (u - x) \);

(K) For each \( x \) such that \( x \not\in F(x) \), there exist an element \( p \) in the topological dual \( E' \) of \( E \) and an open neighborhood \( U(x) \) of \( x \) such that \( \langle p, y \rangle > 0 \) for all \( y \in F(z) - z \) and \( z \in U(x) \).

Then there exists a fixed-point \( x^* \in X \) of \( F \), \( x^* \in F(x^*) \).

**Proof**: Assume that there is no fixed-point of \( F \). Then by considering \( G : X \rightarrow E \) in Theorem 7 to be the identity map, condition (i) in this theorem implies (i) in Theorem 7. It is also easy to see that condition (K) for \( F \) implies condition (ii), so that we have \( x^* \in F(x^*) \cap G(x^*) \) a fixed-point of \( F \), a contradiction.

**REFERENCES**


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